Fourier Analysis 04-20

Review.

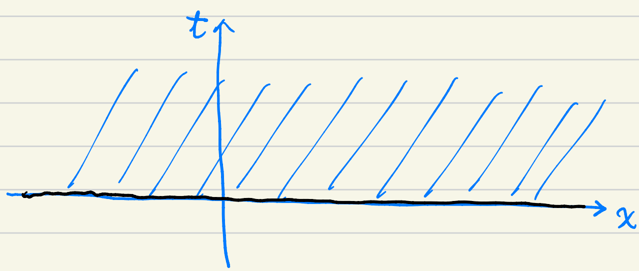
- Heat equation on the real line.

Thm (Uniqueness).

Suppose $u = u(x,t)$ satisfies the following conditions:

① $u \in C(\overline{\mathbb{R}\times\mathbb{R}_+}) \cap C^2(\mathbb{R}\times\mathbb{R}_+)$,

where $\overline{\mathbb{R}\times\mathbb{R}_+} = (-\infty,\infty)\times[0,\infty)$.



② $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R}\times\mathbb{R}_+$

③ $u(x,0) = 0, \quad x \in \mathbb{R}$

④ $u(\cdot, t)$ belongs to $S(\mathbb{R})$ uniformly in t.

Then $u(x,t) \equiv 0$ on $\mathbb{R}\times\mathbb{R}_+$

(A proof using the energy method)

$E(t) = \int_{\mathbb{R}} |u(x,t)|^2 dx, \quad E'(t) \leq 0$

- Steady-state heat equation on the upper half plane.

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & x \in \mathbb{R},\ y > 0 \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

A solution:

$$\hat{u}(\xi, y) = \hat{f}(\xi)\, e^{-2\pi|\xi| y}$$

$$u(x,y) = f * \mathcal{P}_y(x),$$

where $\mathcal{P}_y(x) = \dfrac{1}{\pi} \dfrac{y}{x^2+y^2}$

(Poisson kernel on the upper half plane)

Thm. Let $f \in S(\mathbb{R})$ and $u(x,y) = f * \mathcal{P}_y(x)$.

Then ① $u \in C^2(\mathbb{R} \times \mathbb{R}_+)$ and $\Delta u = 0$

② $u(x,y) \rightrightarrows f(x)$ as $y \to 0$

③ $\int |u(x,y) - f(x)|^2\, dx \to 0$ as $y \to 0$

④ $u(x,y) \to 0$ as $|x| + y \to \infty$

Thm 1 (Uniqueness)

Let $u(x,y) \in C^2(\mathbb{R}\times\mathbb{R}_+) \cap C(\overline{\mathbb{R}\times\mathbb{R}_+})$.

Suppose $\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \text{ on } \mathbb{R}\times\mathbb{R}_+. \\ u(x,0) = 0 \end{cases}$

Moreover, suppose $u(x,y) \to 0$ as $|x| + y \to +\infty$.

Then $u(x,y) \equiv 0$ on $\mathbb{R}\times\mathbb{R}_+$.

<u>Remark</u>: The assumption $u \to 0$ "at infinity" Can not be dropped.

For example: If letting $u(x,y) = y$, ...

Lemma 2 (Mean value property of harmonic functions)

Let Ω be an open set in $\mathbb{R}^2$. Let

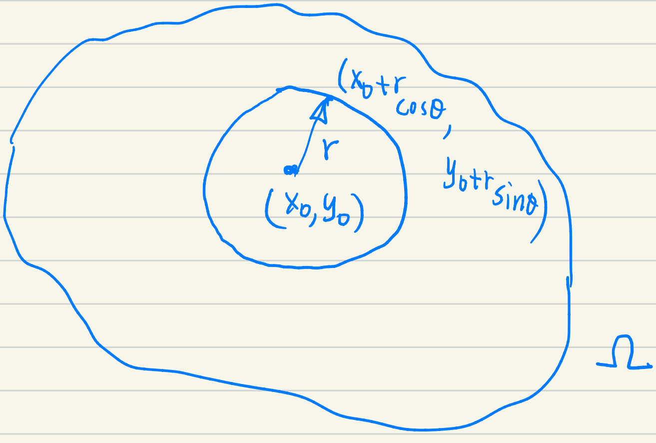
$u \in C^2(\Omega)$. Suppose $B_R(x_0, y_0) \subseteq \Omega$

$\Delta u = 0$

where $B_R(x_0, y_0) := \{(x, y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq R^2\}$

Then $\forall\ 0 < r < R$,

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta,\ y_0 + r\sin\theta)\, d\theta$$



Proof: We write

$$U(r,\theta) = u(x_0 + r\cos\theta,\ y_0 + r\sin\theta),$$

where $0 < r < R$, $0 \leq \theta \leq 2\pi$

We also have

$$\Delta U = 0$$

However,

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

Hence $$r^2 \frac{\partial^2 U}{\partial r^2} + r\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial \theta^2} = 0$$

$$r\frac{\partial}{\partial r}\left(r \cdot \frac{\partial U}{\partial r}\right) + \frac{\partial^2 U}{\partial \theta^2} = 0$$

Now define

$$F(r) = \frac{1}{2\pi}\int_0^{2\pi} U(r,\theta)\, d\theta, \quad 0 \leq r < R.$$

$$r \cdot \frac{d}{dr}\left(r \cdot \frac{dF}{dr}\right) = \frac{1}{2\pi}\int_0^{2\pi} r \cdot \frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) d\theta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^2 U}{\partial \theta^2}\, d\theta$$

$$= -\frac{1}{2\pi} \left.\frac{\partial U}{\partial \theta}\right|_0^{2\pi}$$

$$= 0$$

Hence $\frac{d}{dr}\left(r\frac{dF}{dr}\right) = 0$

$\Rightarrow$ $r\frac{dF}{dr} = \text{const}.$

as $r \to 0$, $\frac{dF}{dr}$ is bdd

So $r\frac{dF}{dr} = 0$

i.e $\frac{dF}{dr} = 0 \Rightarrow F$ is a constant

Hence $F(r) = F(0) = \frac{1}{2\pi}\int_0^{2\pi} U(0,\theta)\, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(x_0, y_0)\, d\theta$$

$$= u(x_0, y_0)$$

$\blacksquare$.

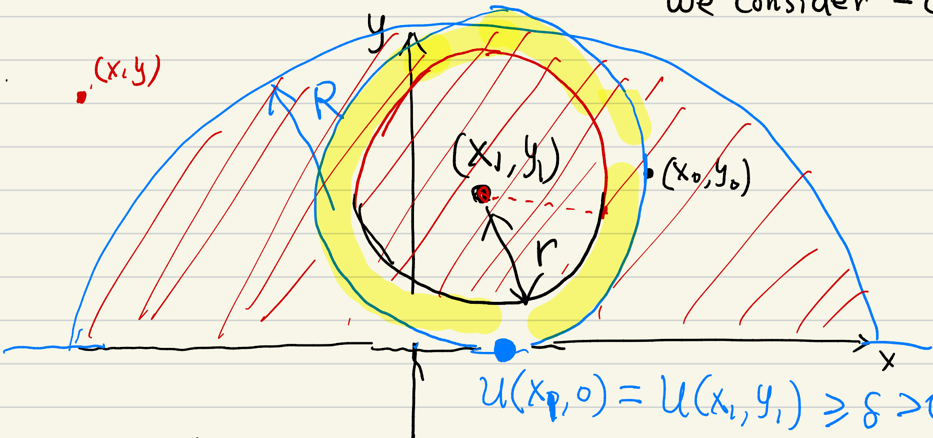
Proof of Thm 1.:

Suppose on the contrary that $u \neq 0$

Then $\exists \ (x_0, y_0) \in \mathbb{R} \times \mathbb{R}_+$ such that

$$u(x_0, y_0) \neq 0.$$

WLOG, assume $\delta := u(x_0, y_0) > 0$. (Otherwise we consider $-u$)



$u(x_p, 0) = u(x_1, y_1) \geq \delta > 0$

Since $u(x, y) \to 0$ as $|x| + y \to +\infty$,

we can find a large $R > 0$ such that

$$u(x, y) < \frac{\delta}{2} \quad \text{outside } B_R(0,0)$$

Now, u is cts on $B_R(0,0) \cap \overline{\mathbb{R} \times \mathbb{R}_+}$

Hence u obtains the maximum at (x_1, y_1) in $B_R(0,0) \cap \overline{\mathbb{R} \times \mathbb{R}_+}$.

$$u(x_1, y_1) \geq u(x_0, y_0) = \delta.$$

By the mean value property

$$u(x_1, y_1) = \frac{1}{2\pi} \int_0^{2\pi} u(x_1 + r\cos\theta,\ y_1 + r\sin\theta)\, d\theta$$

It follows that

$$u(x_1, 0) = u(x_1, y_1) \geq \delta > 0.$$

This leads to a contradiction. $\boxed{}$